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The response of a floating ice sheet to an accelerating line load

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The two-dimensional response of a thin, floating sheet of ice to a line load that accelerates from rest at $t=0$ to a uniform velocity V for $t \geqslant T$ is determined through an integral-transform solution of the linearized equations of motion. If $T=0$ – i.e. if the load is impulsively started with velocity V – the solution exhibits singularities at $V=c_0$, the shallow-water-gravity-wave speed, and $V=c_{\min}$, the minimum speed for transverse motion of the ice, but these singularities are avoided by the acceleration of the load through the critical speeds.

1. Introduction

The problem of a load moving with a uniform velocity V over a sheet of ice floating on a large body of water has been studied both analytically and experimentally; see Squire *et al.* (1996) and the references given therein. The analytical solution is characterized by singularities associated with $V=c_0$, the shallow-water-gravity-wave speed, and $V=c_{\min}$, the minimum speed for transverse wave motion of the ice. These singularities may be avoided through the incorporation of either dissipation (Squire *et al.* 1996, §§4.2.2, 5.6) or nonlinearity (Părău & Dias 2002), but it appears to have been overlooked that they also can be avoided by allowing for acceleration of the load through the critical speeds, Miles (1960).

Against this background, we consider here the wave motion induced by the line load

$$p(x,t)=F\delta[x-X(t)] \tag{1.1}$$

and posit the resulting transverse displacement of the ice in the form

$$\eta(x,t)=\int_0^t G[x-X(\tau),t-\tau]\,\mathrm{d}\tau, \tag{1.2}$$

where δ is Dirac's delta function, G is a Green's function (see §2), and

$$X(t)=\begin{cases} X_A(T)+V(t-T) & (t \geqslant T) \\ X_A(t) & (0 \leqslant t \leqslant T), \end{cases} \tag{1.3}$$

which describes an accelerated motion $X_A(t)$ from rest at $t=0$ to a uniform velocity V for $t \geqslant T$. ($X_A(t)=\frac{1}{2}At^2$ for a uniform acceleration A.) It then follows from the linearity of the boundary-value problem for η that

$$\eta(x,t)=\begin{cases} \eta_A(x,T)+\eta_V[x-X(T),t-T] & (t > T) \\ \eta_A(x,t) & (0 \leqslant t \leqslant T), \end{cases} \tag{1.4}$$

where: η_A is given by (1.2) with $X(t)=X_A(t)$; η_V is given by (1.2) with $X(t)=Vt$ and describes the motion associated with an abruptly imposed velocity V.

2. Boundary-value problem

The two-dimensional, linearized boundary-value problem for the determination of the velocity potential $\phi(x, y, t)$ and the displacement $\eta(x, t)$ of a thin (negligible inertia) sheet of ice that overlies a body of water of depth H is described by Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad (-\infty < x < \infty, -H < y < 0), \tag{2.1}$$

the kinematic boundary conditions

$$\phi_y = 0 \quad (y = -H), \quad \phi_y = \eta_t \quad (y = 0), \tag{2.2}$$

the initial conditions

$$\phi = \eta = 0 \quad (t = 0), \tag{2.3}$$

and the dynamical boundary condition

$$\rho(\phi_t + g\eta) + D\partial_x^4 \eta = -p(x, t) \quad (y = 0), \tag{2.4}$$

where ρ is the density of the fluid, $-\rho(\phi_t + g\eta)$ is the fluid pressure, D is the flexural rigidity of the ice, and p is the externally imposed pressure.

Introducing the Fourier–Laplace transforms

$$[\Phi, N, P] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \int_0^{\infty} \mathrm{e}^{-st} [\phi, \eta, p]\, \mathrm{d}t, \tag{2.5}$$

we obtain

$$\Phi_{yy} - k^2 \Phi = 0 \quad (-H < y < 0), \tag{2.6}$$

$$\Phi_y = 0 \quad (y = -H), \quad \Phi_y = sN \quad (y = 0), \tag{2.7}$$

and

$$s\Phi + (g + Dk^4/\rho)N = -P/\rho, \tag{2.8}$$

the solution of which yields

$$\Phi = sN(k \sinh kH)^{-1} \cosh k(y + H), \tag{2.9}$$

and

$$N = -(P/\rho) k \tanh kH (s^2 + k^2 c^2)^{-1}, \tag{2.10}$$

where

$$c^2(k) = \left(\frac{g}{k} + \frac{Dk^3}{\rho} \right) \tanh kH. \tag{2.11}$$

Transforming the concentrated, moving load (1.1), we obtain

$$P = F \int_0^{\infty} \exp[-s\tau - \mathrm{i}kX(\tau)]\, \mathrm{d}\tau, \tag{2.12}$$

the substitution of which into (2.10), followed by the invocation of the convolution theorem for the inverse-Laplace transform of $P/(s^2 + k^2 c^2)$, yields (cf. (1.2))

$$\eta(x, t) = -\frac{F}{2\pi\rho} \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\tanh kH}{c(k)} \exp\{\mathrm{i}k[x - X(\tau)]\} \sin[kc(t - \tau)]\, \mathrm{d}k. \tag{2.13}$$

If $X(t)=Vt$, the τ integral in (2.3) is elementary, and

$$\eta(x,t)=-\frac{F}{4\pi\rho}\int_{-\infty}^{\infty}\frac{\tanh kH}{kc(k)}e^{ik(x-Vt)}\left[\frac{e^{ik(c+V)t}-1}{c+V}+\frac{e^{ik(V-c)t}-1}{c-V}\right]dk, \qquad (2.14)$$

which is equivalent to Schulkes & Sneyd (1988, equation (2.7)).

3. Numerical results

In order to obtain detailed results from (2.13) numerical methods are necessary. This equation can be written in the form

$$\eta(x,t)=-\frac{F}{2\pi\rho}\mathscr{F}_k^{-1}Q(k,t), \quad Q(k,t)=\frac{\tanh(kH)}{kc(k)}\int_0^t e^{-ikX(\tau)}\sin[kc(t-\tau)]\,d\tau, \quad (3.1)$$

where $\mathscr{F}_k^{-1}$ represents the inverse Fourier-transform operator with respect to the variable k. Numerical calculation of $Q(k,t)$ is not altogether straightforward. In computing the Fourier inverse a large range of k values must be used, which requires the integration of rapidly oscillating functions. Special methods can be used, but here we consider the simpler case in which the acceleration (or deceleration) of the source is uniform. Then $Q(k,t)$ can be expressed in terms of Fresnel integrals as follows.

For an accelerating load $X=\frac{1}{2}At^2$, where A is a constant, the integral in (3.1) can be written in the form

$$\frac{1}{2i}[I(c)e^{ickt}-I(-c)e^{-ickt}]$$

where

$$I(c)=(k/\beta)\exp(i\gamma^2)[F((\beta(t+\alpha)))-F(\gamma)].$$

Here $\alpha=c/A, \gamma=(\frac{1}{2}kA)^{1/2}$, and the function $F(x)$ is defined by setting

$$F(x)=\int_0^x \exp(it^2)\,dt$$

and can be defined in terms of Fresnel integrals,

$$F(x)=\sqrt{\pi/2}(C(x\sqrt{2/\pi})-iS(x\sqrt{2/\pi}),$$

in the notation of Abramowitz & Stegun (1964). For the decelerating load $X(t)=Vt-\frac{1}{2}At^2$ we find

$$I(c)=(k/\beta)\exp(i\gamma^2)[F^*\big((\beta(t+\alpha))\big)-F^*(\gamma)],$$

where the star denotes the complex conjugate, and now $\alpha=(V+c)/A$.

Figures 1(a)–1(d) show the time development of the wave system due to line loads moving with constant (or zero) acceleration. As in Schulkes & Sneyd (1988) we present the results in the form of mesh plots of the surface elevation in the (x,t)-plane. The x-axis is placed in the immediate foreground, and the second horizontal axis represents time. In each case the surface is shown in a frame of reference moving with the load. We use dimensionless units in which the velocity, length and time scales are

$$v_0=c_{\min}, \qquad \eta_0=\frac{F}{\rho c_{\min}^2}, \qquad t_0=\frac{\eta_0}{c_{\min}}, \qquad (3.2)$$

respectively. Here $c_{\min}$ is the minimum phase velocity. For example in experiments carried out by Takizawa (1978) the ice thickness was 0.14 m, with $c_{\min}\approx 5\,\mathrm{m\,s^{-1}}$. The

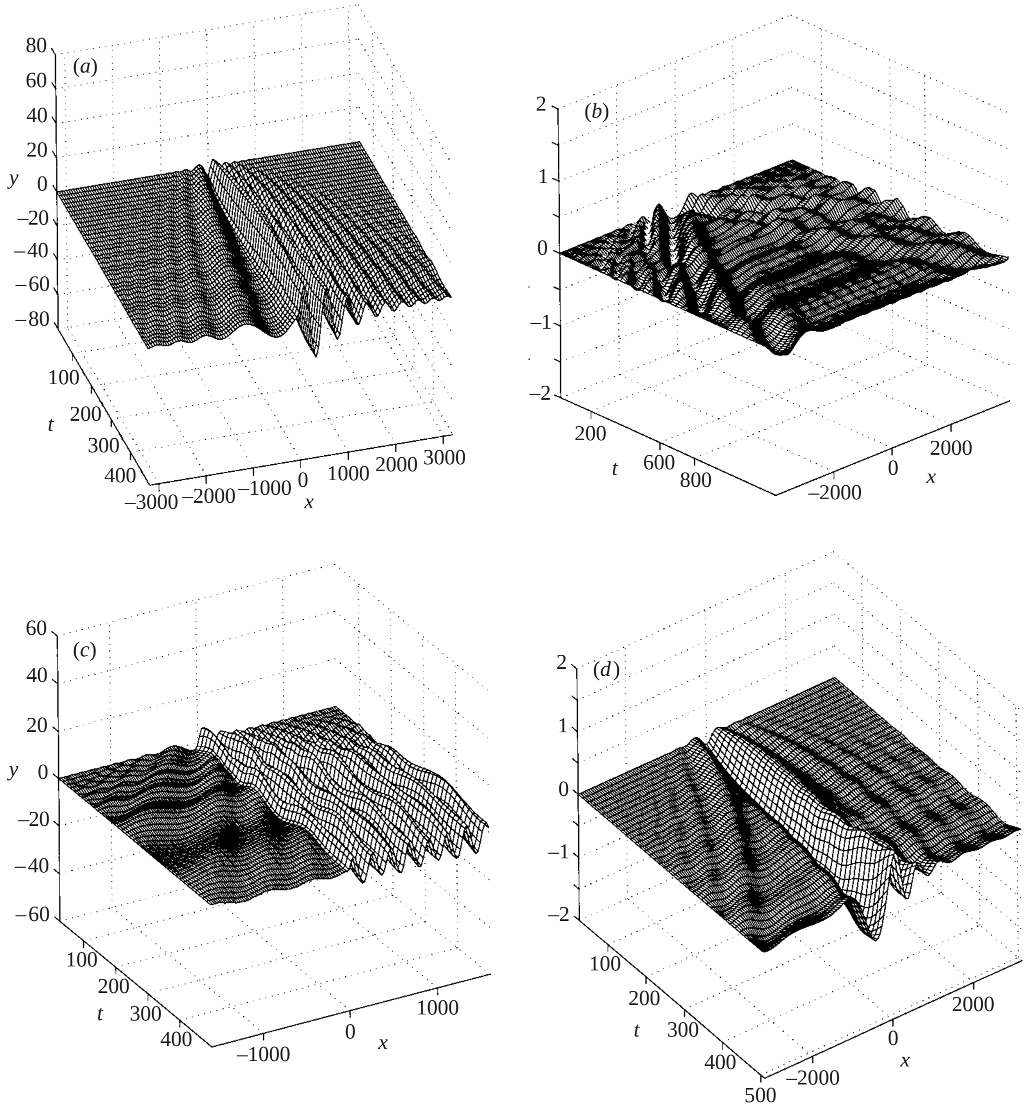

FIGURE 1. Time development of the wave system due to line loads: mesh plots of the surface elevation for (*a*) constant subcritical velocity $v' = 0.4$, (*b*) constant supercritical velocity $v' = 1.17$, (*c*) constant acceleration 0.0784 to a final speed $v' = 2.0$, (*d*) initial speed $v' = 0.4$, then constant deceleration to rest at time $t' = 1275$.

load was 350 kg, and assuming a load width of 1 m the length and times scales would be $\eta_0 = 14$ cm and $t_0 = 0.028$ s. A prime – for example v' – is used to denote the non-dimensionalized variables.

Figure 1(*a*) shows the wave pattern created by a load impulsively accelerated to a subcritical speed $v' < 1.0$. which then continues at constant speed. This is similar to figure 7 of Schulkes & Sneyd (1988). Shorter, faster, waves propagate before the source, and slower, longer, waves behind. Figure 1(*b*) is similar, but now the speed is supercritical with $v' > 1.0$. Here the difference in wavelength between the forward and backward waves is more pronounced, as in figure 8 of Schulkes & Sneyd (1988).

Figure 1(*c*) shows the effect of a load starting from rest, then moving with a constant acceleration $a' = 0.0784$ to a maximum velocity of approximately $2c_{\text{cmin}}$. The most noticeable difference is that the wave amplitude increases more slowly because the load speed is increasing. In figure 1(*d*) the load accelerates impulsively to a speed $0.4c_{\text{cmin}}/2$ then decelerates uniformly, coming to rest at time $t' = 1275$. At this point the waves have radiated away from the load, and the steady-state deflection begins to evolve.

4. Conclusions

In this article we have developed a method of calculating the deflection of a uniform floating ice sheet in response to a load moving at varying speed. Numerical Fourier-transform inversion can be used to visualize the ice-sheet deflection. We find no noticeable effect as a load accelerates through the critical speed c_{cmin}. This is to be expected since the displacement due to a load travelling at *constant* speed c_{cmin} grows linearly with time (Schulkes & Sneyd 1988). For loads moving with non-constant acceleration special methods may be necessary to evaluate rapidly oscillating integrals.

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